

8

Current



Fig.8.1.1, ref.¹. A lighthouse at night in a thunderstorm helps the ships to navigate

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8.1 Introduction

It is evident that the concept of current is of extreme importance for electrical engineers as well as for all other human beings. All electric circuits are driven by a current flow and electric energy is transported from power plants to the consumer through cables.

At the consumer the current is used to drive electric circuits, to drive electric motors and to give light. Earlier, light emission from incandescent bulbs was one of the most important applications of current. Use of light for traffic control and navigation at sea are just some of many important applications. Nowadays, light is produced much more efficiently using fluorescent tubes or LED's.

The development of our modern civilization was very much based on current.

The telephone and the telecommunication system was based on a flow of relatively small currents in very long wires across oceans and continents. The radio transmitter and receiver use electromagnetic waves across long distances over sea and land and therefore no wire connection but the equipment could not be constructed without current flow in circuits. We find current flow everywhere even in animals and human beings, which could not function without current flow in nerves.

Current is a flow of charge through a conductor such as a metal wire. In metals and in superconductors the charge flow is due to a flow of electrons. In 8.2 we get a short overview of how electrons move inside a metal if we have no E-field and if we have an E-field inside the metal. If we have an E-field, the electrons will be displaced with the so-called drift velocity v_d . This implies a current through the metal wire. The current density J is defined in 8.3. In 8.4 we find a relation between v_d and the current density J in the wire. Ohm's law states that the E-field is proportional to current density J . This is discussed for a wire in 8.5. The proportionality constant between E and J is the resistivity ρ . In 8.10 we look at the temperature dependence of ρ for different types of conductors. Section 8.7 describes the energy levels for electrons in insulators, semiconductors and conductors. Section 8.8 describes the E-field inside a wire, which is connected to a voltage source. In 8.9 we deal with the equivalent diagram and power relations for a battery. The electrochemical function of a battery is described in chapter 9.

8.2 How an electron moves inside a metal

If we consider an *element* with atomic number Z it means that the atom has Z protons in its nucleus and Z electrons in the electron cloud outside the nucleus. Some elements are able to form a metal crystal structure if many atoms are brought together. In such a metal crystal structure, some of the electrons, the so-called conduction electrons will be able to move through the whole metal crystal structure while the other electrons are bound to the atoms.

If we consider a single atom with Z electrons, it may deliver one electron to the common crystal structure. This electron becomes a conduction electron that can move through the metal crystal structure almost without resistance. The other $Z-1$ electrons cannot move away from the atom. They are bound to the atom and they can only orbit around the atomic *nucleus*.

If we have N atoms in the metal crystal structure, there will be N positively charged atomic *nuclei*, that cannot move around in the crystal. The atomic nuclei have to stay in fixed positions. However, they can vibrate with some amplitude, and these vibrations give the metal crystal its temperature which is proportional to the kinetic energy of the vibrations. It is the whole atom (the nucleus and the $Z-1$ electrons) that vibrates. The conduction electron lives its own life and surfs around in the whole metal crystal structure on the equipotential surfaces.

It should be noted that some elements may deliver more than one conduction electron (e.g. 2 electrons), when the atoms form metal crystals.

As it might be seen from the above, the positive charges (the nuclei) cannot move around in a metal crystal structure. It is only the negatively charged conduction electrons that can move around. Below, we will study how such electrons move around in the metal crystal structure.

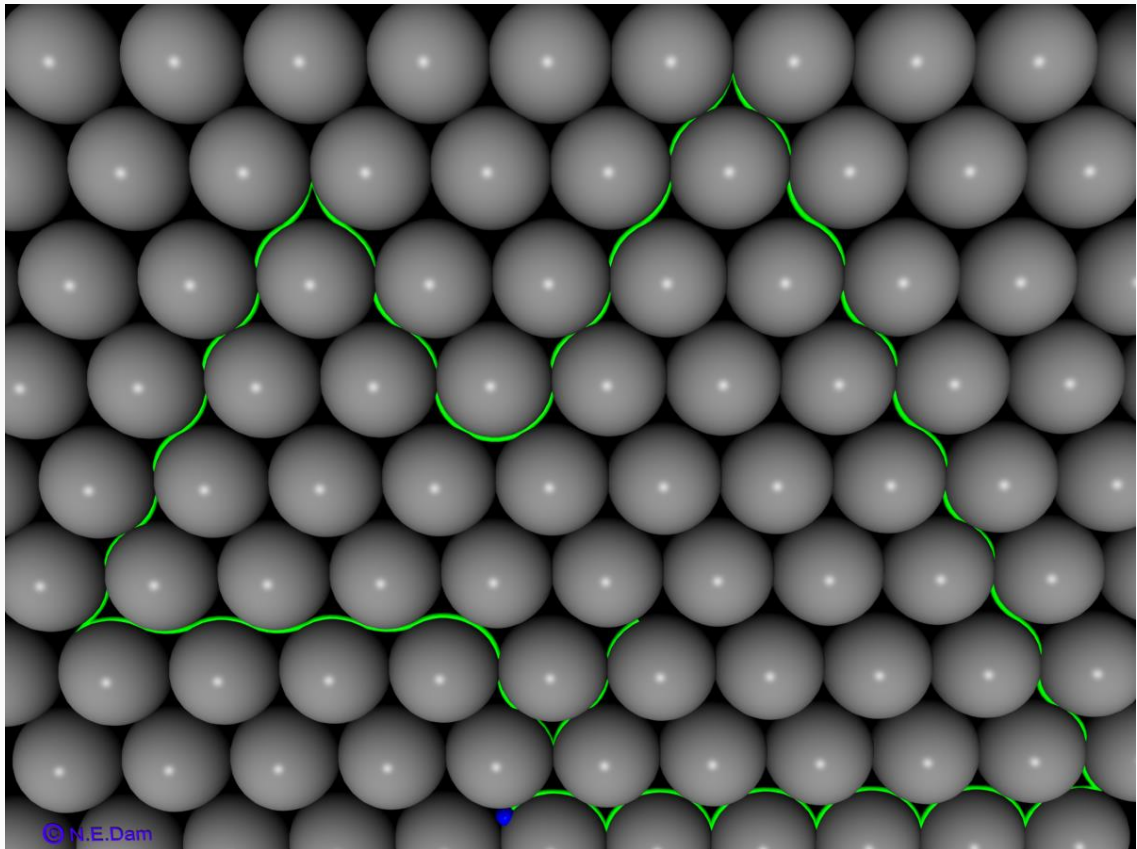


Fig.8.2.1, (N.E. Dam ©). A single conduction electron on its way through a metal crystal structure.

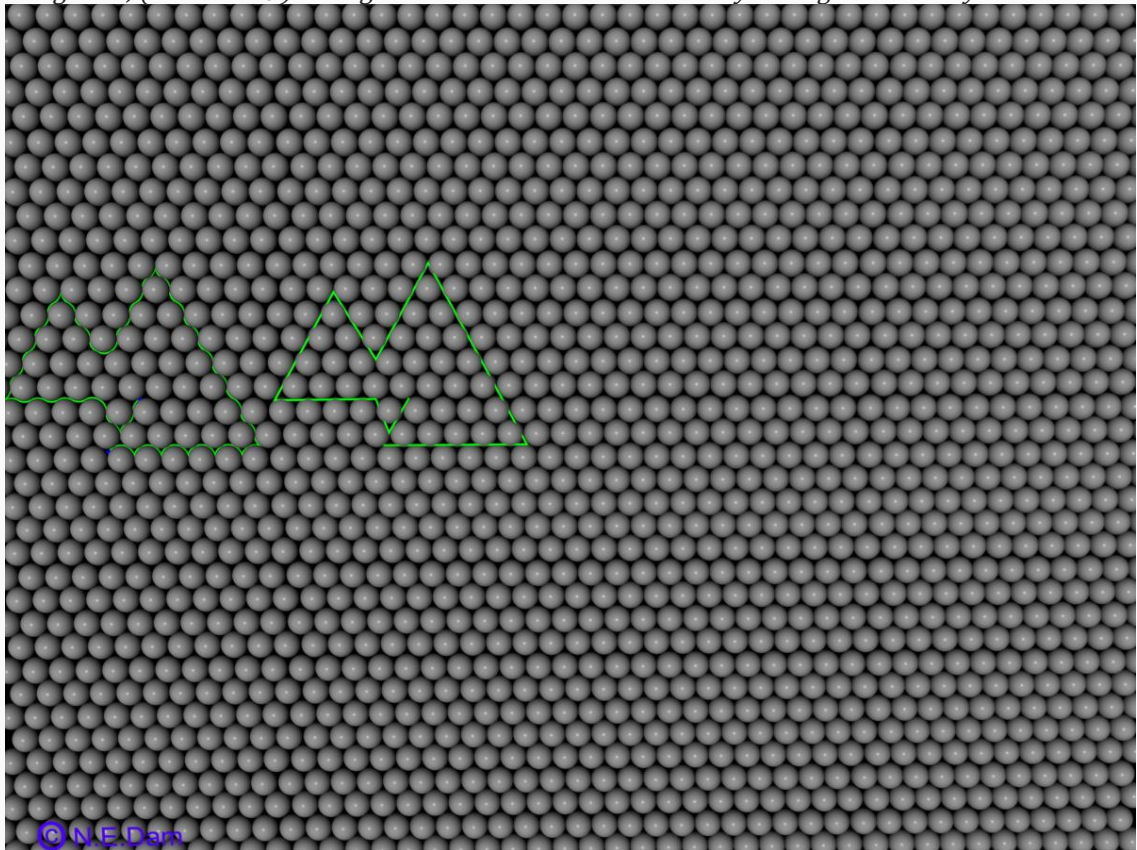


Fig.8.2.2, (N.E. Dam ©). On a larger scale one barely sees the small ripples along the electron path where the electron follows the equipotential surfaces of the individual atoms.

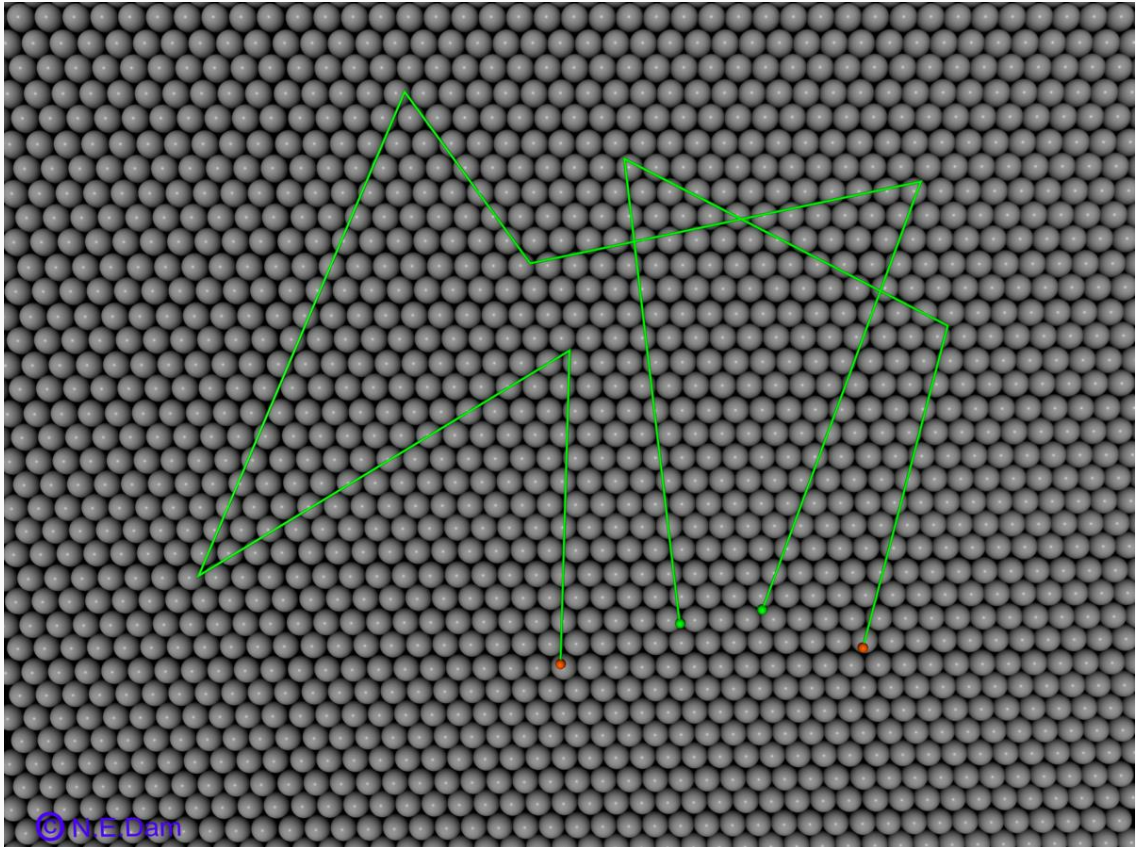


Fig.8.2.3, (N.E. Dam ©). The random motion gives no net current in any direction as shown in a simplified way for two conduction electrons.

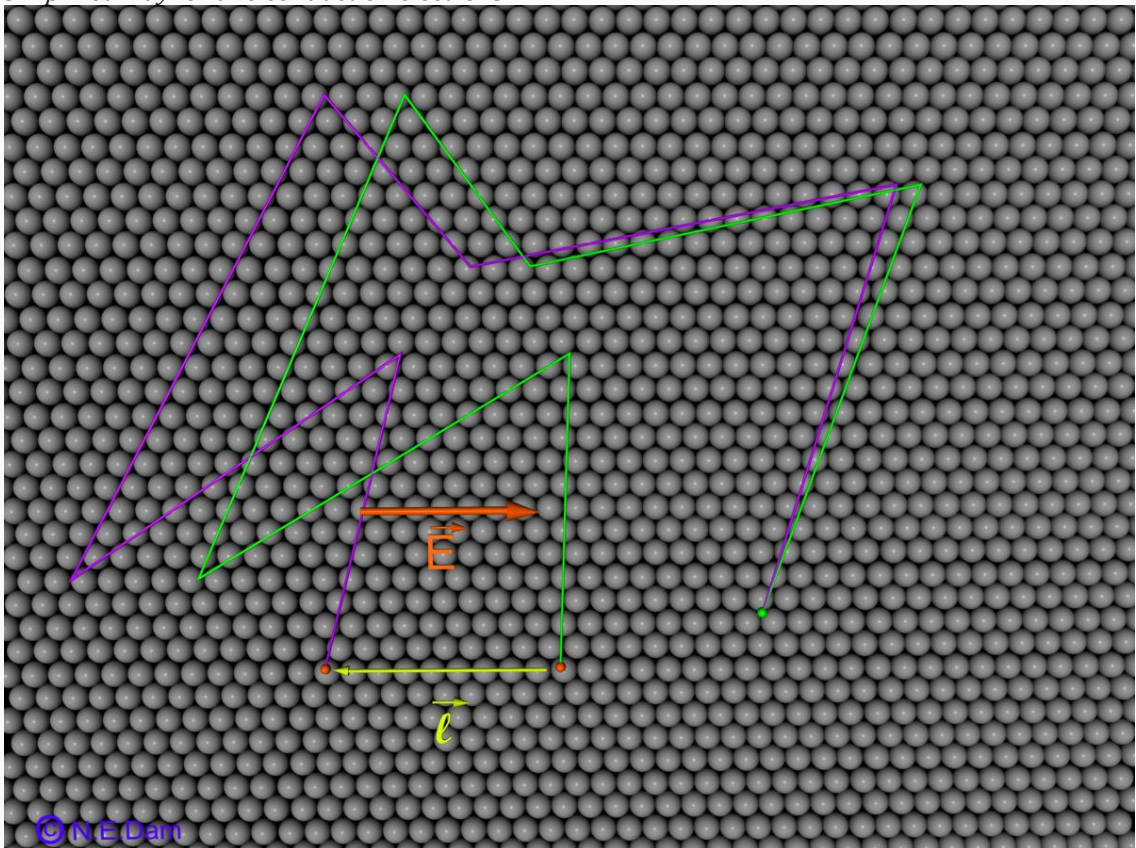


Fig.8.2.4, (N.E. Dam ©). If we have $E = 0$ inside the metal (the electrostatic condition) the electron follows the green path from the green point to the orange point. If we have an electric field $E > 0$ inside the metal, the electron follows the purple path.

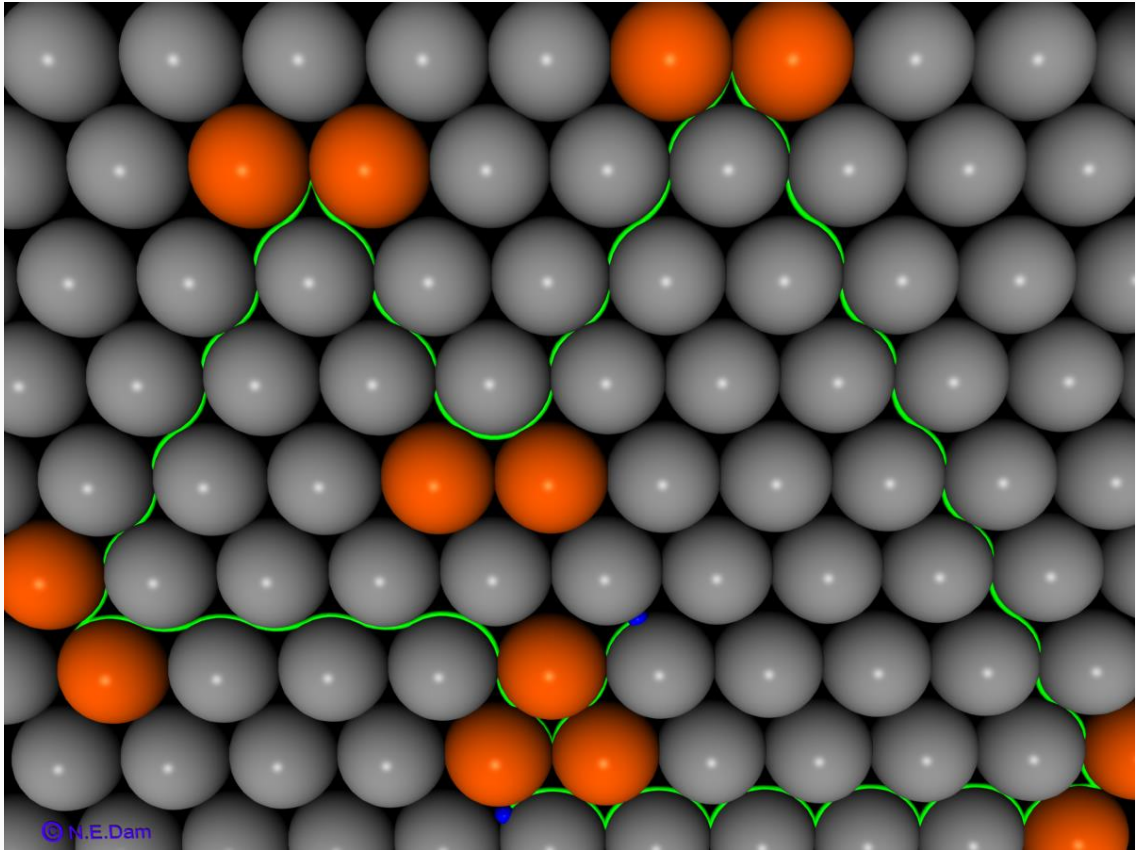


Fig.8.2.5, (N.E. Dam ©). Each time a conduction electron collides with an atom, there will be an exchange of energy between the electron and the atom in the crystal.

Fig.8.2.1 shows a single conduction electron on its way through a metal crystal structure. A metal is the most important type of conductor. The electrons follow the equipotential surfaces around the atoms because the electric force acting on an electron is typically in the order of 10^{20} times stronger than the gravitational force from the Earth!

The reader may show this using Coulomb's law with the distance $r \approx 10^{-10}$ m and Newton's second law $F = m a$, where $a = g = 9,81 \text{ m/s}^2$.

A conduction electron moves on equipotential surfaces in random directions and changes direction when it collides with atoms in the crystal structure.

It looks like the conduction electron is moving in a 2D plane. In fact it moves in random directions in 3D. However, it is difficult to show this 3D movement in a figure because the electron would move out of the 2D plane and become invisible behind the atoms in the 2D plane.

On a larger scale, where we cannot see the individual atomic equipotential surfaces, it looks like the electrons move along straight lines as shown in fig.8.2.2.

The two green paths are identical except that the left one has ripples and the right one does not. Hence, we consider a conduction electron to move along straight lines changing direction when the electron collides with the atoms in the crystal structure. A real electron may travel a distance corresponding to thousands or millions of atoms before it collides with an atom. If we should zoom out to see the path of an electron, we would not be able to see the individual atoms and hence the path would look like straight lines connected at points where the electron changes its direction.

In electrostatics, the E field is zero within the conductor and there is no net current. The electrons are moving randomly along straight lines in all directions with a speed of about 1000 km/s! This random motion gives no net current in any direction as shown in fig.8.2.3 for two conduction electrons. They start at time $t = t_0$ at the two green points. At time $t = t_0 + \Delta t$ we find them at the two orange points. One electron has a net movement to the left during the time Δt and the other electron has a net movement to the right during the same time Δt . If we consider an average of many electrons, there will be no net transport of charge in any direction.

If the E field is not zero there is a net current. It is explained below:

We now assume, that the metal crystal is a part of a metal wire and the wire is connected to an external battery with the positive pole connected to the left side of the metal wire and the negative pole connected to the right side of the wire. We thereby get an E-field from left to right inside the metal wire as shown in fig.8.2.4.

Fig.8.2.4 shows one of the two conduction electrons from the previous figure.

If we have $E = 0$ inside the metal (the electrostatic condition) the electron follows the green path from the green point to the orange point.

If we have an electric field $E > 0$ inside the metal, the conduction electron follows the purple path because the electron is affected by a force in the opposite direction of the E-field. In the figure there is a drift to the left. Hence, the electron will end up at a point to the left of what would have been without an E-field and likewise for all other conduction electrons. Therefore, there will be a net transport of negative charge to the left corresponding to a positive current to the right if we have an E-field to the right.

On average, the conduction electrons end up at a distance $\vec{\ell}$ to the left of what would have been without the E-field. In this way, the negative charge has been displaced $\vec{\ell}$ to the left during the time interval Δt .

We define the so-called *drift velocity* by

$$\vec{v}_d = \frac{\vec{\ell}}{\Delta t} \quad \text{eq. 8.2.1}$$

A typical magnitude may be in the order of 0,1 mm/s.

When we turn on a switch, the current flows immediately. The information about the field in the conductor propagates with the velocity of light (much faster than 1000 km/s, but slower than 300 000 km/s). It is like a company of soldiers. The "start" command from the sergeant reaches the soldiers' ears at the speed of sound, which is much faster than the marching speed.

Often the moving conduction electron collides with another electron belonging to an atom in the crystal lattice. The conduction electron then changes its direction just like a ball rolling on a billiard table rebounds when it hits the sides. At the same time, the moving conduction electron exchanges energy with the atoms in the crystal lattice as shown in fig.8.2.5. The orange atoms in the figure may exchange energy with the moving conduction electron. The orange atoms will thereby change their vibrational energy.

If we have no E-field ($E=0$) in the crystal structure the energy exchange between the conduction electrons and the (orange) atoms will not change the total vibrational energy of the crystal structure. The temperature of the crystal is related to the vibrational energy of the atoms in the crystal structure. Hence, the temperature will remain constant in this situation.

However, if we have an E-field ($E > 0$) inside the crystal structure, we have a net flow of negative charge in the opposite direction of the E-field. In this case, the conduction electrons deliver more energy to the (orange) atoms than they receive from the (orange) atoms. The total vibrational energy of the crystal structure will increase and hence the temperature will increase. The vibrations of the individual (orange) atoms will propagate to the other (grey) atoms.

The electric field implies an electric force on an electron. This force accelerates the electron and the electron acquires extra kinetic energy. The moving conduction electron then transfers kinetic energy to the atoms, where the moving conduction electron is reflected in other directions as shown in fig.8.2.5. Then the atoms in the crystal lattice vibrate more intensely. The increasing kinetic energy of the vibrating atoms causes the temperature of the crystal to increase. This is why the metal is heated (Joule heat) when we have a net current through the metal. Often it is an unwanted waste of heat (e.g. electric bulbs), but sometimes it is desired (electric heaters). Repeating some of the most important facts from the above description of the conduction electrons movement, we have:

$E = 0$ (Electrostatic condition)

On a large scale relative to the atom size, the conduction electrons move in a zig-zag motion along straight lines. The average position of such electrons does not change and they receive as much energy from the crystal lattice as they deliver to the crystal lattice when they collide with atoms in the crystal lattice. The speed of this motion is in the order of 10^6 m/s.

$E > 0$ (Electrodynamic condition)

The conduction electrons still move in a zig-zag motion along straight lines as explained above. However, because of the electric force on the conduction electrons, the average position of such electrons changes with the so-called drift velocity (in the order of 0,1 mm/s) and such electrons deliver more energy to the crystal lattice than they receive when they collide with atoms in the crystal lattice. This explains the energy loss and the resistance in the metal.

The E-field propagates with about 10^8 m/s through the metal.

8.3 Definition of the current density J

The current density J is the current per unit area.

$$J \equiv \frac{I}{A} \quad \text{eq. 8.3.1}$$

In this equation, the area should be perpendicular to the current direction. We may consider the current as a flux of the current density, i.e. as a J-flux

$$I = \Phi_J = J A \quad \text{eq. 8.3.2}$$

which could be compared with the E-flux defined in chapter 3

$$\Phi_E = E A \quad \text{eq. 8.3.3}$$

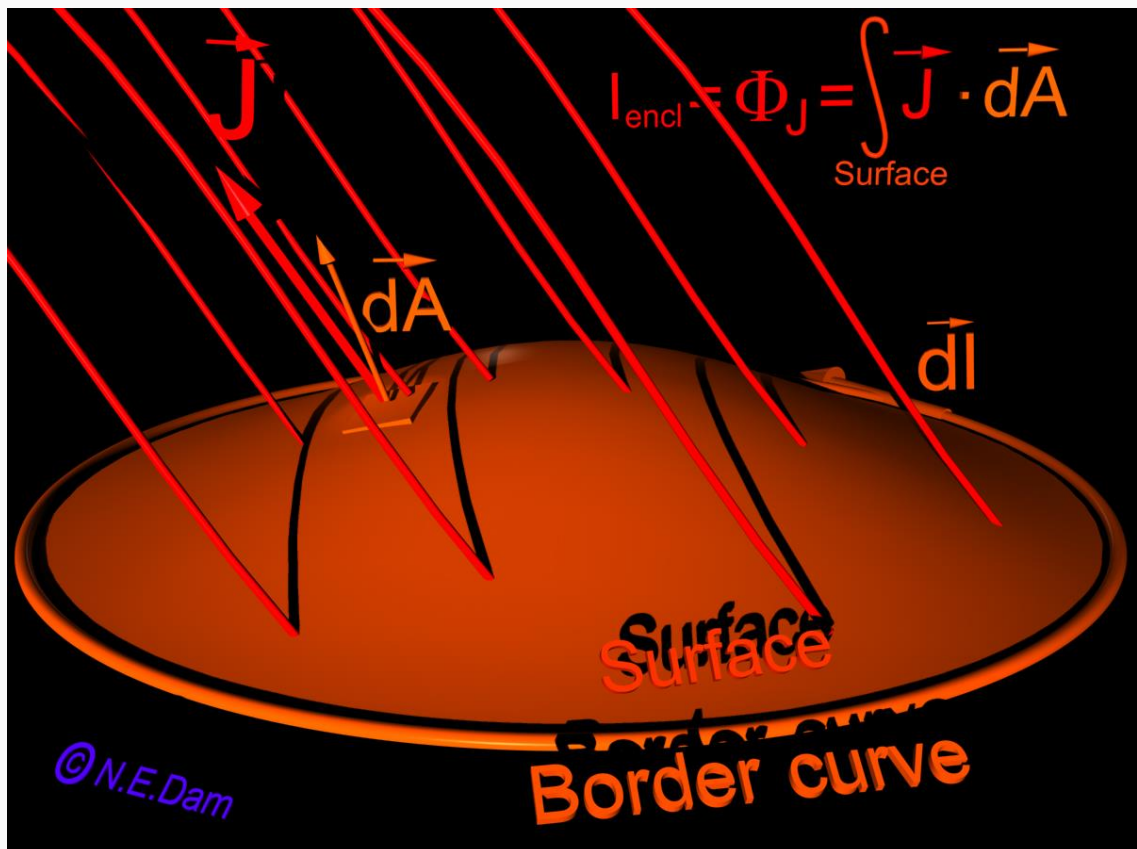


Fig.8.3.1, (N.E. Dam ©). The flux of the current density through a surface equals the current.

The J-flux through a curved surface is calculated quite analogous to the E-flux

$$I = \Phi_J = \int_A \vec{J} \cdot \vec{dA} \quad \text{eq. 8.3.4}$$

Fig.8.3.1 shows J flux lines through a curved surface. E.g. each flux line may represent 1 A. In total we count 9 flux lines through the surface. Hence the current or the flux is 9A. If the surface was more perpendicular to the flux lines, the flux would probably

be higher as we remember from the introduction of the E-flux through a tilted area. The J-flux through the small area is $\vec{J} \cdot \vec{dA}$, and the J-flux through the curved surface is given by eq.8.3.4.

The current density vector $\vec{J}$ has the same direction as the J-flux lines or - more correctly - $\vec{J}$ has the same direction as the tangent at any point on a flux line or flux curve. The magnitude of the current density vector is proportional to the density of J-flux lines.

Please note that the current is a scalar, not a vector. The direction of the current at a point is given by the direction of vector $\vec{J}$. It is analogous to the fact, that the E-flux through a surface dA is a scalar, because it is calculated as the scalar product $\vec{E} \cdot \vec{dA}$.

8.4 The relation between v_d and J

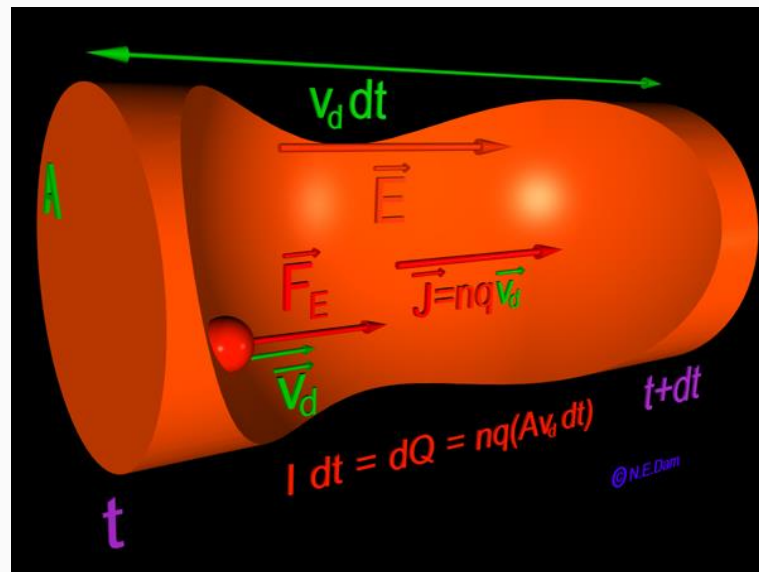


Fig.8.4.1, (N.E. Dam ©). Derivation of the relation between v_d and J .

The current can be considered as one of the two following possibilities.

1. Positive charge moving in the same direction as the E-field
2. Negative charge (electrons) moving opposite to the E-field

In an electrolyte (a substance with moving positive and negative charges, often a liquid) both no. 1 and no. 2 is active. In metals and in superconductors only no. 2 is active!

The positive current direction is the direction of positive charge flow, which is the same as the direction of the E-field.

Therefore we prefer to consider no. 1 in theoretical calculations even if we know, that no. 2 describes the physical reality in metals.

Fig.8.4.1 shows a positive charge moving through a metal wire in the same direction as the E-field. The charge velocity is the drift velocity v_d .

The current magnitude I is the amount of charge dQ passing through an area A per unit time.

$$I = \frac{dQ}{dt} \text{ eq. 8.4.1}$$

Then $dQ = Idt$ is the amount of charge passing through the surface A during the time interval dt . This amount of charge can be calculated in another way using v_d :

The charge travels the distance $v_d dt$ during the time dt . The shown charge is moving from end surface A at time t to the opposite end surface A at time $t+dt$. Hence all the charge inside the cylinder (a section of the wire) is moving out of the cylinder during the time dt . The volume of the cylinder is $A v_d dt$. The density of charges is n . It is the *number* of charges per volume. Then $n(A v_d dt)$ is the number of charges inside the

cylinder. One single charge has the charge q . Hence $dQ = nq(Av_d dt)$ is the total charge within the cylinder. This dQ must equal the amount of charge passing through the end surface A during the time interval dt :

$$Idt = nqAv_d dt \Rightarrow I = nqAv_d \text{ eq. 8.4.2}$$

Then for the magnitude of the current density we get:

$$J \equiv \frac{I}{A} = nqv_d \text{ eq. 8.4.3}$$

Since the direction of J is the same as the direction of $\vec{v}_d$, we have finally:

$$\vec{J} = nq\vec{v}_d \text{ eq. 8.4.4}$$

Note that $\vec{J}$ and $\vec{E}$ have the same direction.

8.5 Ohm's law for a wire

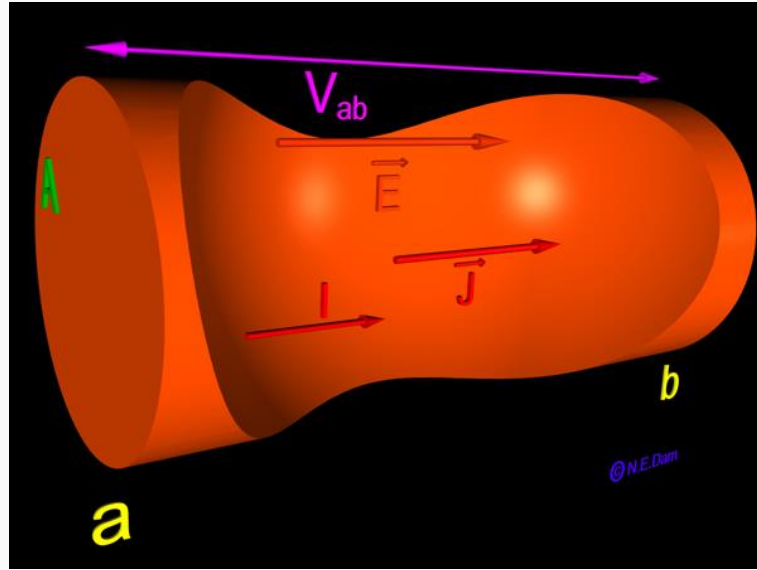


Fig.8.5.1, (N.E. Dam ©). Derivation of the resistance for a metal wire.

Fig.8.5.1 shows a metal wire. The cross section area is A and the length is L . If we connect the positive pole from a battery to the left end surface (a) and the negative pole to the right end surface (b), there will be a potential drop V_{ab} from a to b, and the E-field magnitude will be

$$E = \frac{V_{ab}}{L} \quad \text{eq. 8.5.1}$$

The unit is V/m. Ohm's law states, that *the electrical resistivity* is defined as

$$\rho = \frac{\vec{E}}{\vec{J}} = \frac{E}{J} \quad \text{eq. 8.5.2}$$

The unit is Ωm . Note, that we can only have a fraction of two vectors if the vectors are parallel. In this case they are co-parallel (in the same direction) as noted at the end of section 8.4.

Using eq.8.5.1 and eq.8.5.2 we get

$$\rho = \frac{E}{J} = \frac{\frac{V_{ab}}{L}}{\frac{I}{A}} = \frac{V_{ab}}{I} \frac{A}{L} = R \frac{A}{L} \quad \text{eq. 8.5.3}$$

Then for the resistance R of the wire we get:

$$R = \rho \frac{L}{A} \quad \text{eq. 8.5.4}$$

The resistance is proportional to the length L and inversely proportional to the cross sectional area A . The proportionality constant is the electrical resistivity ρ .

The constant ρ allows comparison between different materials, since ρ is independent of L and A .

If we depict V as a function of I we get a curve $V(I)$. If this curve is a straight line with constant slope $R = V/I$ we say, that the material obeys Ohm's law. This is the case for a normal conductor and also for a homogeneous semiconductor (doped or not doped), but not for a diode or a semiconductor device.

Note, that in case of nonohmic material, we still have $R = V / I$ and $R = \rho L / A$

In this case, R depends on V and ρ depends on E .

The *electrical conductivity* is defined this way

$$\sigma = \frac{1}{\rho} \quad \text{eq. 8.5.5}$$

The unit is $1/(\Omega\text{m}) = \Omega^{-1}\text{m}^{-1} = \text{S/m}$.

This means that the resistance R of the wire becomes

$$R = \rho \frac{L}{A} = \frac{L}{\sigma A} \quad \text{eq. 8.5.6}$$

The unit is Ω .

It is recommended to solve Problem 8.1 now.

8.6 E-field inside a wire

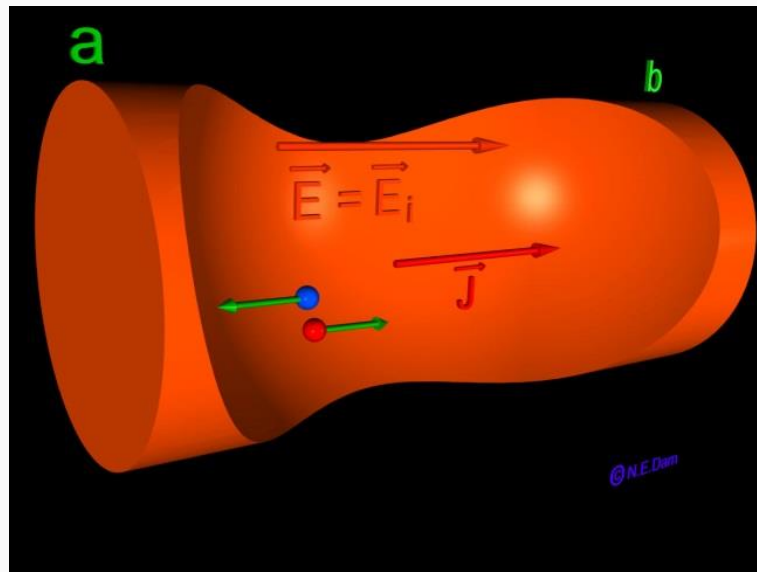


*Fig.8.6.1, (N.E. Dam ©).
A piece of wire with an E-field.*

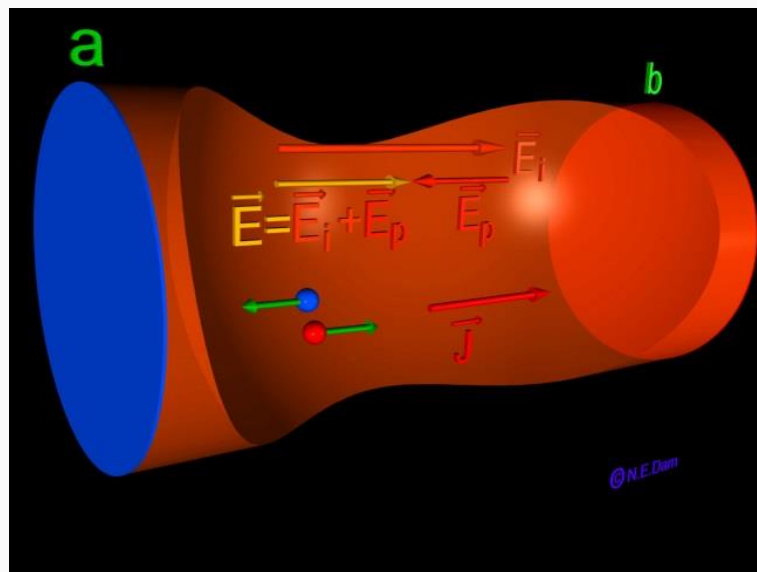


*Fig.8.6.2, (N.E. Dam ©). The same piece of wire as
above with an E-field and with polarization
charges at the end surfaces.*

When we connect a battery to a wire, there will be a potential drop along the wire. The E-field is in the wire direction and its magnitude is the potential drop divided by the length of the wire.

If we move a piece of a wire in a magnetic field, it is possible to generate an E-field inside the wire, E_i , without a battery connected to the wire. This will be explained in detail in chapter 13.

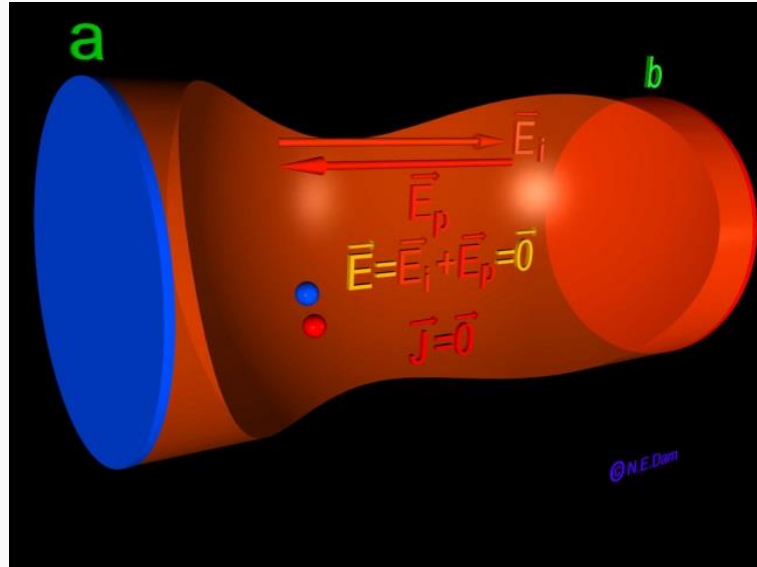


Fig.8.6.3, (N.E. Dam ©). The same piece of wire as above with an E-field and with even more polarization charges at the end surfaces.

The following explains why the current stops flowing, when a wire is cut over and we have an E-field in the wire to begin with. This initial E-field E_i may be generated by moving the wire in a magnetic field.

To begin with the induced E-field, E_i , will give an electric force F_E on both negative and positive charges. This will cause the charges to move towards the end surface at a and b as shown in fig.8.6.1. In fact we separate the positive and negative charges. In normal metals and superconductors only negative charges are moving. Charge will now be accumulated at a and b as shown in fig.8.6.2. The polarization charges at a and b will cause a new E-field, E_p , which is opposite to E_i . Now the resulting E-field, E , will be smaller than before and hence the forces on the charges and the current will also be smaller.

After a short time the accumulated charge will increase to a level, where $E=0$, $F_E=0$ and $J=0$ as shown in fig.8.6.3. The current stops flowing.

8.7 The equivalent diagram for a battery

Discharging the battery

Fig.8.7.1 shows an equivalent diagram for a battery with a load R . The internal resistance R_i is due to the energy loss when the ions are moving through the electrolyte. The positive current is flowing out of the $+$ terminal through the load R and back to the $-$ terminal. Since we have a potential drop across R_i from left to right, the output terminal voltage is

$$V = V_{\text{Bat}} - R_i I \quad \text{eq. 8.7.1}$$

This relation is shown in fig.8.7.2. For $I=0$ the output voltage is the so-called *electro-motive-force*, the *emf* of the battery

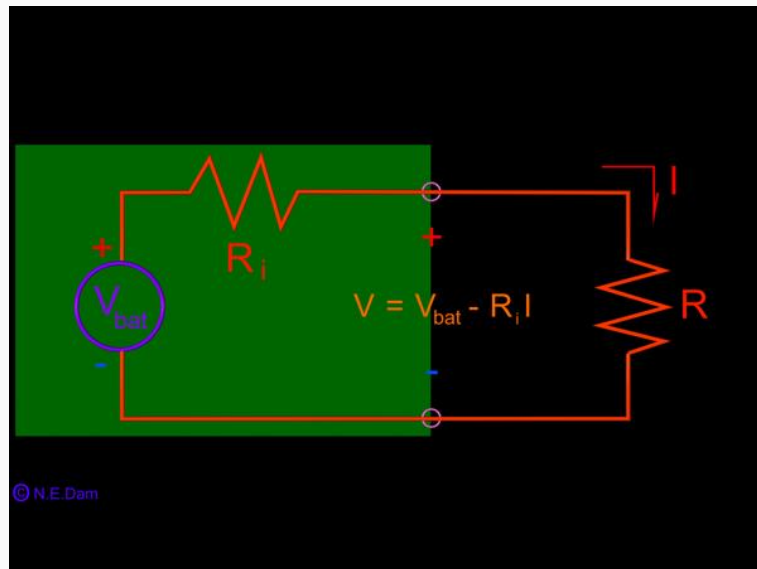


Fig.8.7.1, (N.E. Dam ©). Electric equivalent for a battery with a load R .

$$V_{\text{Bat}} = V_{\text{emf}} = \mathcal{E} \quad \text{eq. 8.7.2}$$

The electro-motive-force is *not* a force. It is the output voltage from the unloaded battery.

For $I=I_{\text{sc}}$ (short circuit across the output terminals) we have $V=0$.

The slope of the straight line is $-R_i$. This slope may be found as the slope between the two points $(0, V_{\text{Bat}})$ and (I, V) . Hence, we do not have to make a short circuit in order to find the internal resistance.

Fig.8.7.3 shows the change in potential in a circuit with a battery, an internal and an external resistance.

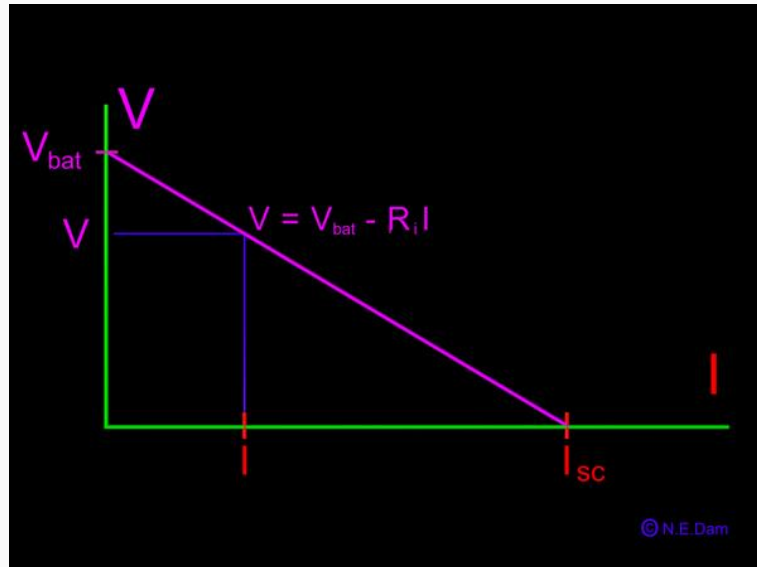


Fig.8.7.2, (N.E. Dam ©). Output voltage from a battery.

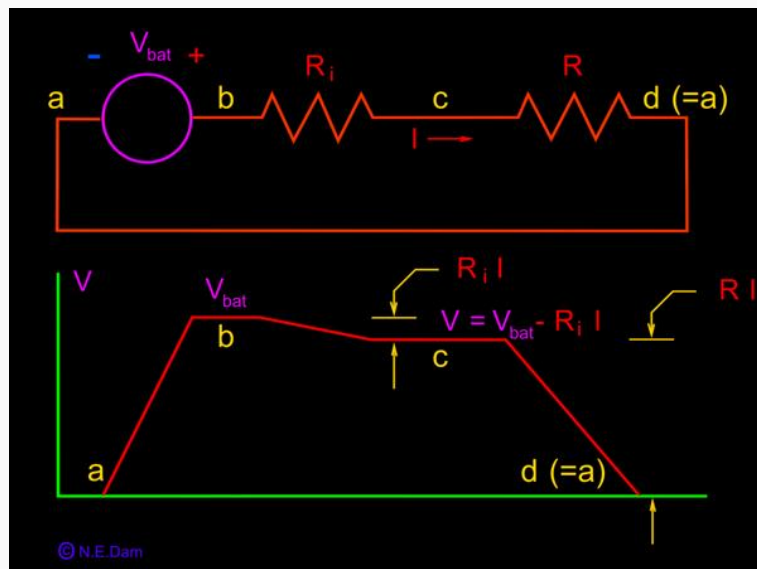


Fig.8.7.3, (N.E. Dam ©). Potential changes inside a circuit with a battery, an internal resistance and an external resistance.

Charging the battery

Fig.8.7.4 shows the equivalent diagram when charging the battery using an external emf. We have that

$$V_{\text{Bat}} < V_{\text{ext}} \quad \text{eq.8.7.3}$$

so the current flows in the correct direction to charge the battery as shown in the figure. Now we have a potential drop across R_i from right to left. Hence the output terminal voltage is

$$V = V_{\text{Bat}} + R_i I \quad \text{eq.8.7.4}$$

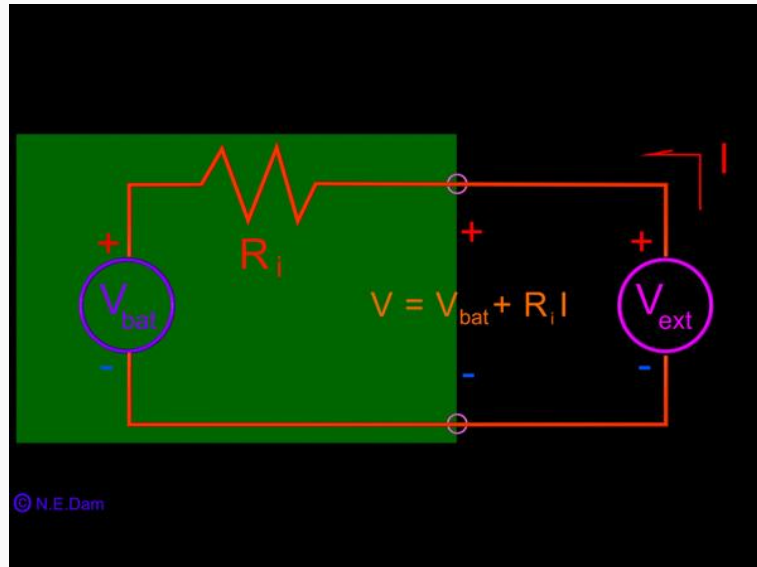


Fig.8.7.4, (N.E. Dam ©). Charging a battery.

Power relations

When discharging

If we multiply eq.8.7.1 by I , we get

$$VI = V_{\text{Bat}}I - R_i I^2 \quad \text{eq. 8.7.5}$$

The first term on the right side is the power output from the chemical cells. From this we subtract the power dissipated in the battery and get the total power output.

When charging

If we multiply eq.8.7.4 by I , we get

$$VI = V_{\text{Bat}}I + R_i I^2 \quad \text{eq. 8.7.6}$$

The first term on the right side is the power delivered to the chemical cells. To this we add the power dissipated in the battery and get the total power input.

8.8 Energy levels for electrons

In this section we discuss how to classify the different types of materials based on the so-called band structure, i.e. the energy levels of the electrons. There is also a brief explanation of conductivity in semiconductors.

It is well known that the electrons in an atom occupy the energy levels in so-called shells corresponding to different values of the principal quantum number, n .

The smallest value of n is 1. The corresponding shell is denoted the K-shell. In this shell there can only be a maximum of two electrons.

The next value of n is 2. The corresponding shell is denoted the L-shell. In this shell there can be a maximum of 8 electrons.

Higher values of n are shown in fig.8.8.1. The vertical scale is the total energy of the electrons for $Z = 1$ (the element Hydrogen). The energy levels depend on n as shown by the formula in the figure.

The K-shell has only one possible level for the potential energy of the electrons. Higher shells (higher n values) have more possible levels.

The electrons occupy the levels from the lowest level (lowest potential energy) and upwards. A much more detailed description of the energy levels for the electrons is given in a chapter named “Atoms”, which is not a part of this book.

If N atoms are together in a single crystal, the so-called bands are formed. If the atoms K and L shells are filled with electrons, there will be a total of N times 2 of the K-electrons and N times 8 of the L-electrons. Due to a quantum mechanics condition, there can only be 2 electrons in each energy level (one electron with spin up and one electron with spin down). As a result, N levels are formed, each with two K-electrons. These N levels are occupied with the 2 N electrons from the K-shells. The N energy levels are very close together and form a so-called band, in this case a K-band. This is illustrated in fig.8.8.1. Likewise, the L-band has the 8 N electrons from the L-shells.

Fig.8.8.2 shows the principle for the *band structure* in different materials.

The insulator

We assume that the K, L and M bands are fully occupied with electrons. Conduction would be possible if one of the “highest” electrons (one of the electrons with the highest potential energy) in the M band could be excited to the empty N band. Such an electron would be free to move around in the crystal as a conduction electron. This is, however, not possible because the distance (on the energy scale, the distance is called the energy gap) from the M-band to the N-band is too large. Hence, the electron has to stay in the M-band. Since the M-band is filled, the M-electrons are strongly bound to the nucleus. The total potential energy for the M-electrons are much lower when they stay together in a filled M-band. The electrons are not allowed to move away from the individual atoms.

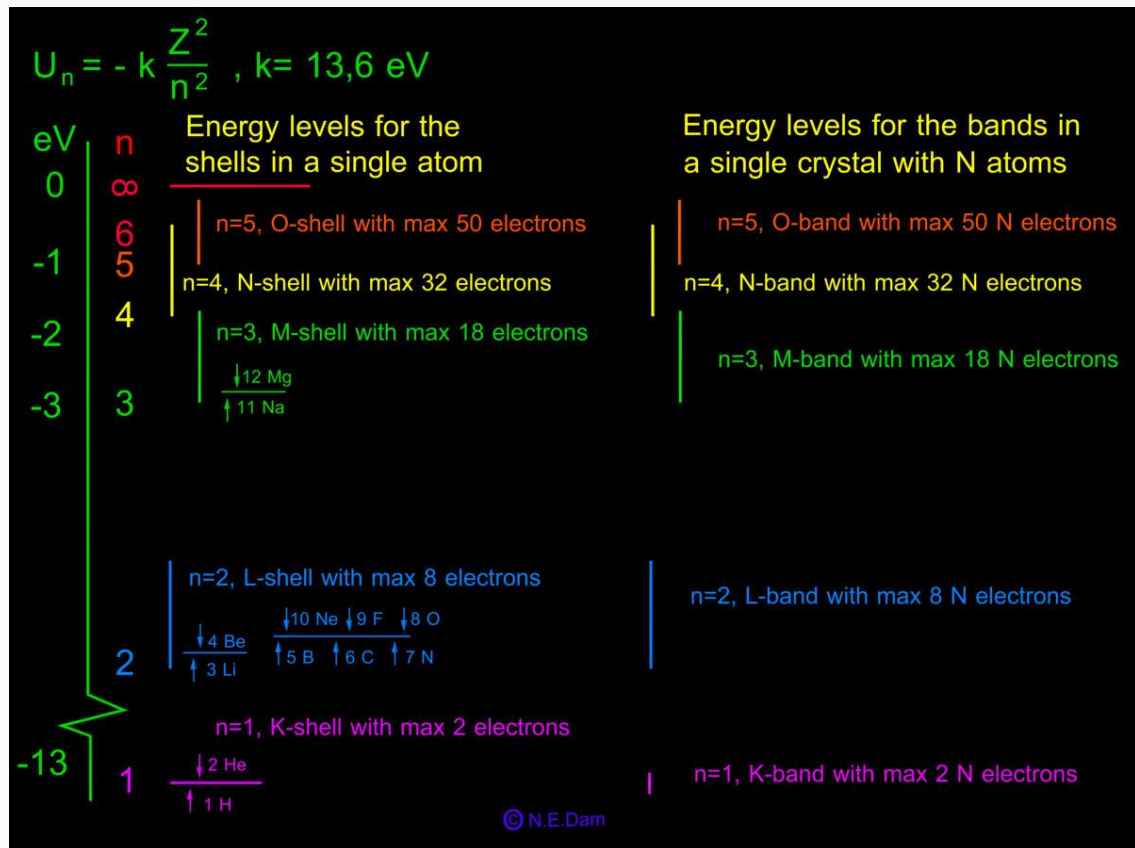


Fig.8.8.1, (N.E. Dam ©). In a single atom, the electrons are arranged in so-called shells. In a single crystal, the electrons are arranged in so-called bands.

The natural (intrinsic) semiconductor

The energy gap is not as large as for the insulator. This means that sometimes it is possible to excite an electron from the top of the M-band to the bottom of the N-band. The N-band becomes a conduction band. The excitation is due to thermal vibrations. Therefore, if the temperature increases, more electrons will be transferred to the N band and the conductivity will increase. Hence, the resistivity decreases when the temperature increases at relatively low temperature values as shown in fig.8.9.4. However, at relatively high temperature values, the resistivity increases due to the more intense vibrations of the atoms in the crystal lattice as explained for “normal” metal conductors in section 8.2.

The N-type semiconductor

A very pure semiconductor is polluted with a small amount of special impurity atoms (e.g. 1 impurity atom per 1 million atoms). This introduces an extra energy level with an extra electron just below the N-band. This electron can easily be excited to the N-band, the conduction band. In this case the conduction is due to *negatively* charged electrons in the conduction band. Hence, the name N-type.

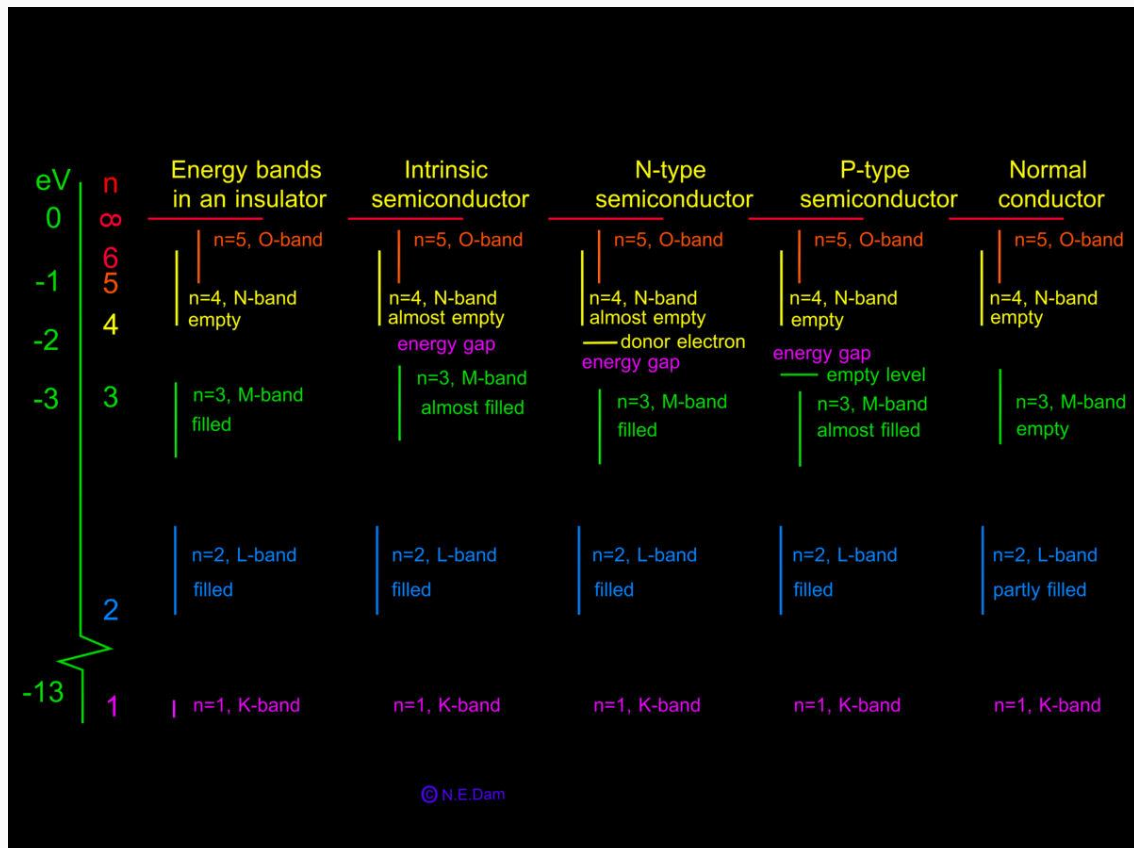


Fig.8.8.2, (N.E. Dam ©). The band structure for different materials.

The P-type semiconductor

A very pure semiconductor is polluted with a small amount of special impurity atoms. This introduces an extra empty energy level just above the M-band. One of the electrons in the top of the M-band can now easily be excited to the extra empty energy level. This creates a deficit of one electron in the top of the M-band. This deficit is called a “hole” and it is *positively* charged. Hence, the name P-type semiconductor. This hole can now move around in the M-band and thereby create conductivity. In fact, it is the other electrons around the hole, that move around. However, it may be treated theoretically as a positive charge with the same mass as an electron moving around in the M-band.

The normal conductor

As an example we may consider a Li atom. It has a filled K-shell with two electrons and a partly filled L-shell with only one electron. In a metal crystal structure of Li-atoms we get a filled K-band and a partly filled L-band. The partly filled L-band becomes a conduction band.

8.9 Temperature dependence of ρ

In this section we consider $\rho(T)$, the electrical resistivity as a function of the temperature. We consider different types of materials such as perfect conductors (e.g. extremely pure and perfect metal crystal structures), normal conductors (e.g. metals with some impurities and crystal lattice imperfections), superconductors and semiconductors. The crystal lattice imperfections are very often due to mechanical treatments (e.g. when the material is bended or rolled during the manufacturing process, “cold working”). Lattice imperfections are to some extent removed if the material is heated to such a high temperature, that new and perfect crystals grow on expense of the imperfect crystals (“annealing” process). The total electrical resistivity is a sum of several contributions:

$$\rho(T) = \rho_{\text{perfect}} + \rho_{\text{impurities}} + \rho_{\text{lattice imperfections}} \quad \text{eq. 8.9.1}$$

The temperature dependence is mainly due to the first term on the right side. This term is the intrinsic resistivity, $\rho_{\text{perfect}} = \rho_i(T)$, the curve is shown in fig.8.9.1. “Intrinsic” refer to the nature of the pure material without any disturbance from impurities and imperfections.

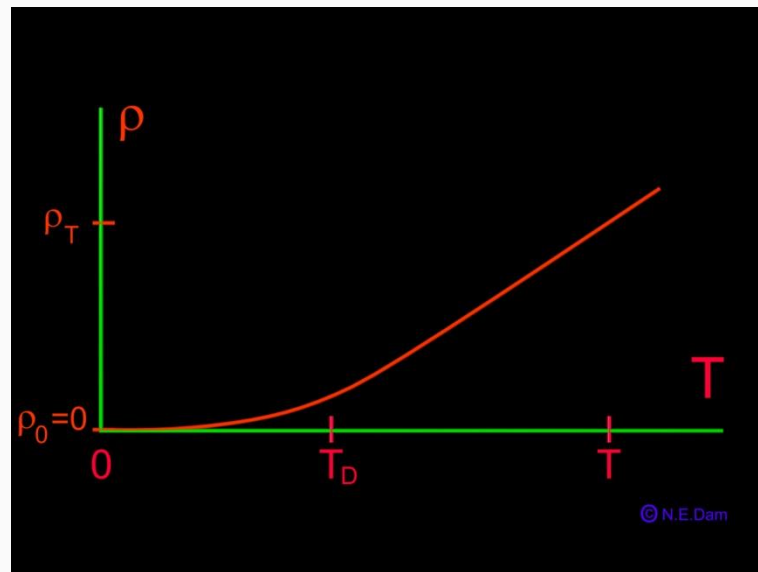


Fig.8.9.1, (N.E. Dam ©). $\rho(T)=\rho_i(T)$ for a perfect conductor.

For the second and third terms on the right side we have:

- The two terms are proportional to the concentration of impurities respectively lattice imperfections.
- They are both relative independent of the temperature.
- The sum of the two terms is the residual resistivity ρ_0

Now, eq.8.9.1 may be written in a shorter way

$$\rho(T) = \rho_i(T) + \rho_0 \quad \text{eq. 8.9.2}$$

This relation is known as Matthiessen’s rule².

Perfect conductor

If we imagine a perfect metallic crystal structure without impurities and without crystal lattice imperfections, we have a perfect conductor, where the resistivity is the intrinsic resistivity. Fig.8.9.1 shows the resistivity as a function of the temperature. The resistivity depends on oscillations of the atoms in the crystal lattice. These oscillations depend on the temperature. At high temperatures, the oscillations have large amplitudes. This increases the probability of collisions between the conduction electrons and the vibrating atoms. These collisions are responsible for the resistivity of the perfect conductor and causes the resistivity to increase when the temperature increases. Above the so-called Debye temperature, T_D , the resistivity increases with the first power of the temperature

$$\rho_i(T) \cong c_1 \cdot T^1, T \gtrsim T_D \quad \text{eq. 8.9.3}$$

Here, c_1 is a constant and T_D could be in the order of 200 K, see ref³.

At relatively low temperatures the resistivity of a perfect conductor goes with the fifth power of the temperature

$$\rho_i(T) \cong c_2 \cdot T^5, T \rightarrow 0 \quad \text{eq. 8.9.4}$$

Here, c_2 is a constant. At very low temperatures (a few kelvin), the resistivity is extremely low, but not exactly equal to zero.

Note, that we cannot cool down to exactly 0 K, but perhaps to 10^{-6} K.

Also note, that a perfect conductor is not the same as a superconductor, where ρ is exactly equal to zero below the so-called critical temperature, T_c .

Normal conductor

When we add impurities and lattice imperfections to a perfect conductor, we get a normal conductor. The word “normal” also indicates, that it is not another kind of conductor (superconductor or semiconductor).

Fig.8.9.2 shows $\rho(T)$ for a normal conductor. The resistivity $\rho(T)$ is almost linear above the Debye temperature T_D . At $T = 0$ K we have the residual resistivity ρ_0 , which is proportional to the concentrations of impurities and imperfections. The residual resistivity ρ_0 is rather independent of the temperature. The difference between a perfect conductor and a normal conductor is the impurities and the lattice imperfections causing the residual resistivity ρ_0 . It should be added to the intrinsic resistivity $\rho_i(T)$ to get the total resistivity for the normal conductor:

- Resistivity for the pure and perfect material (perfect conductor): $\rho_i(T)$
- Additional resistivity due to impurities and lattice imperfections: ρ_0
- Total resistivity for the normal conductor : $\rho(T) = \rho_i(T) + \rho_0$

The effect of adding impurities and lattice imperfections to a perfect conductor is that ρ_0 is added to the curve of $\rho_i(T)$ in fig.8.9.1. Thereby, the curve in fig.8.9.1 is displaced upwards by the amount ρ_0 and we get the curve in fig.8.9.2.

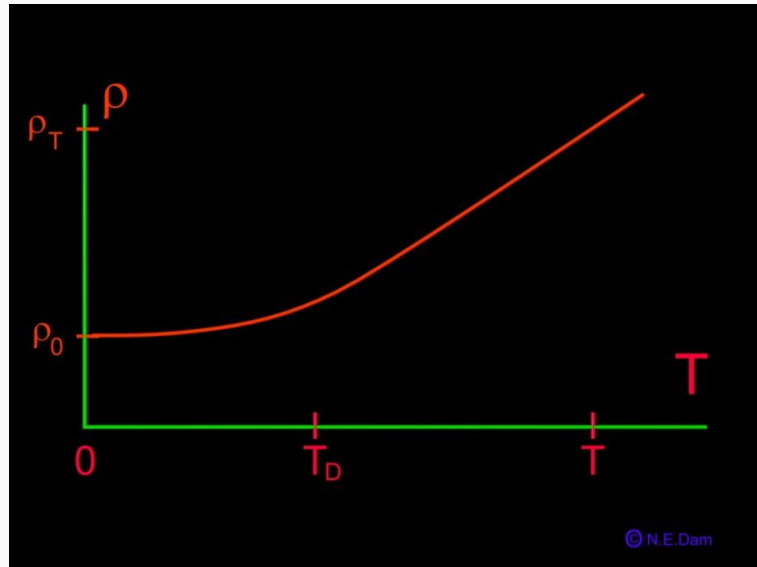


Fig.8.9.2, (N.E. Dam ©). $\rho(T)$ for a normal conductor.

Superconductor

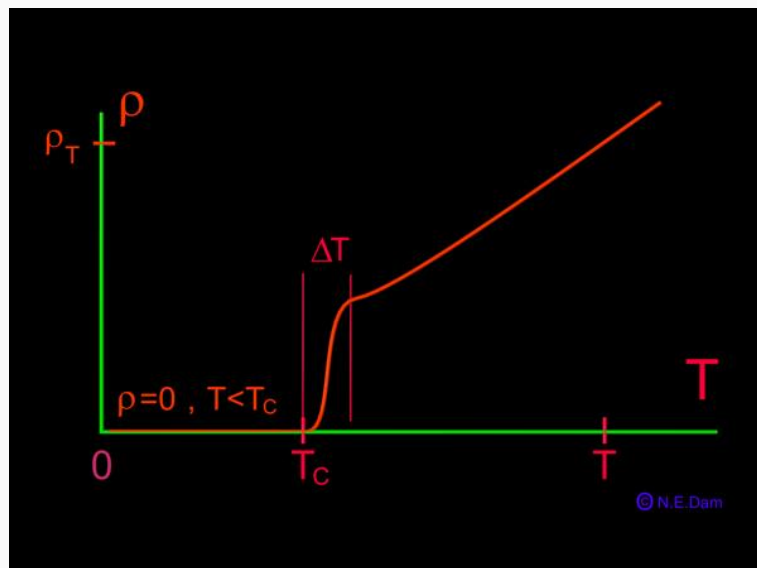


Fig.8.9.3, (N.E. Dam ©). $\rho(T)$ for a superconductor.

Fig.8.9.3 shows $\rho(T)$ for a superconductor. Below the so-called critical temperature T_c , ρ is exactly equal to zero! T_c may be in the order of a few K for the superconducting elements and up to about 23 K for the best superconducting metal alloys. Superconducting ceramic materials may have T_c values up to about 150 K. If we have a very strong magnetic field close to the superconductor, it is possible to suppress the superconductivity so the material will not be superconducting at low temperatures. In that case $\rho(T)$ follows the same curve as for the normal conductor shown in fig.8.9.1. The width of the transition from the normal state to the superconducting state is ΔT . It may be in the order of 1 K.

Semiconductor

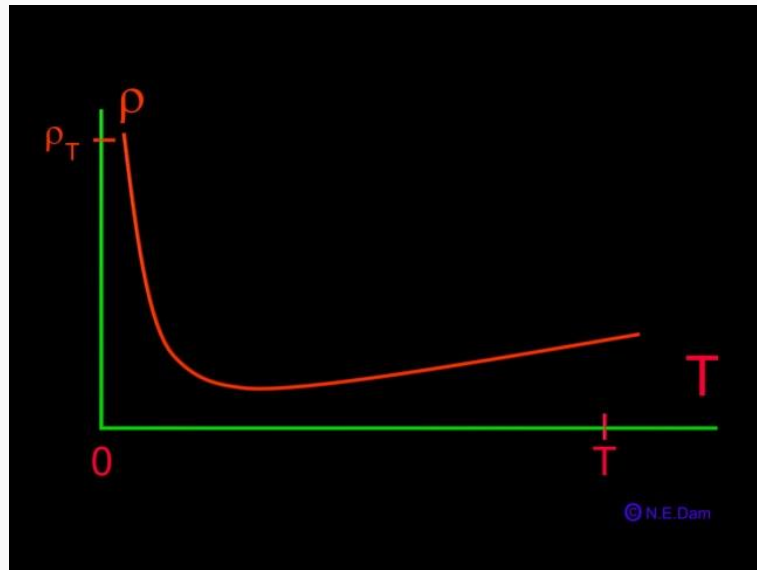


Fig.8.9.4, (N.E. Dam ©). $\rho(T)$ for a semiconductor.

Fig.8.9.4 shows $\rho(T)$ for a homogeneous semiconductor (doped or not doped). In a semiconductor, there are two phenomenons to affect the resistivity in opposite directions:

1. Electrons are excited to the conduction band as explained in section 8.8. This excitation depends on the temperature T . When T increases at relatively low T values, more electrons are excited to the conduction band. This causes higher conductivity and lower resistivity. Hence, $\rho(T)$ is a decreasing function of T at low T values.
2. However, at higher temperature values, the resistivity increases due to the more intense vibrations of the atoms in the crystal lattice as explained for “normal” metal conductors in section 8.2.

These two contributions to the resistivity are always present in the semiconductor. However, contribution no. 2 is almost negligible at low temperatures, where contribution no. 1 dominates.

At higher temperatures, it is contribution no. 2, which dominates.

8.10 Approximation of $\rho(T)$

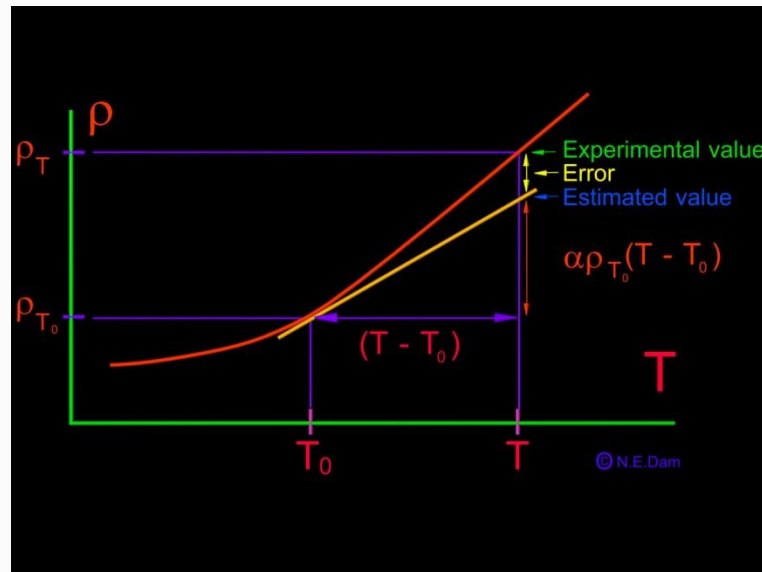


Fig.8.10.1, (N.E. Dam ©). Approximation of $\rho(T)$ for a normal conductor.

Consider a wire of length L and cross-section area A . We assume that the resistance R_{T_0} of the wire or the resistivity ρ_{T_0} of the wire material at some temperature T_0 is known. In the section below we derive a formula, which can be used to find the resistance or the resistivity at a temperature other than T_0 .

We assume that the following are known:

- The resistivity ρ_{T_0} at the temperature T_0 . Often, $T_0 = 293 \text{ K} \cong 20 \text{ C}$.
- The so-called temperature coefficient α .

The temperature coefficient of Copper is $\alpha = 0,00393 \text{ K}^{-1} = 0,393 \% \text{ K}^{-1}$. It means, that if we increase the temperature by 1 K (from 293 to 294 K or from 20 to 21 C), the *relative* increase in resistivity is $\alpha = 0,393 \%$. The resistivity at 293 K is ρ_{T_0} . Therefore, the *absolute* increase in resistivity becomes $\alpha\rho_{T_0}$. Hence, the resistivity at 294 K becomes the resistivity at 293 K (it is ρ_{T_0}) plus the increase in resistivity ($\alpha\rho_{T_0}$) when the temperature increases from 293 to 294 K. It gives $\rho_{T_0} + \alpha\rho_{T_0}$. If we increase the temperature from T_0 to T , the increase in temperature becomes $(T - T_0)$ and the increase in the resistivity due to this temperature increase becomes $\alpha\rho_{T_0}(T - T_0)$. This increase should be added to the resistivity at 293 K which is ρ_{T_0} . In total we get the following resistivity at the temperature T

$$\rho(T) = \rho_T \cong \rho_{T_0} + \alpha\rho_{T_0}(T - T_0) \quad \text{eq. 8.10.1}$$

The above explanation and formula is illustrated in fig.8.10.1. It shows a sketch of $\rho(T)$ for a normal conductor. The curvature of $\rho(T)$ is exaggerated in order to illustrate the approximation more clearly. As explained above, the temperature coefficient α is defined in such a way that $\alpha\rho_{T_0}$ is the increase in resistivity per kelvin and therefore it is the slope of the tangent to the curve at the point (T_0, ρ_{T_0}) .

The linearly approximation in eq.8.10.1 may also be written this way

$$\rho_T \cong \rho_{T_0} \{1 + \alpha(T - T_0)\} \quad \text{eq. 8.10.2}$$

Problem 8.2 shows how to use this approximation. The problem also illustrates the error when we use the approximation. The error is shown in fig.8.10.1 and it is the difference between the estimated value according to eq.8.10.2 and the experimentally determined value on the curve.

We use $R = \rho \frac{L}{A}$ (from eq.8.5.6) and multiply eq.8.10.2 by $\frac{L}{A}$ on both sides:

$$R_T \cong R_{T_0} \{1 + \alpha(T - T_0)\} \quad \text{eq. 8.10.3}$$

Problems

Pr 8.1 Electric field, potential difference, resistance of a wire

The diameter of a copper wire is $d = 1,382 \text{ mm}$

The current is $I = 1,5 \text{ A}$. The length is $L = 100 \text{ m}$.

At $T_0 = 293 \text{ K}$ we have for the resistivity⁴

$\rho_{T_0} = 1,678 \cdot 10^{-8} \Omega \text{ m}$ (uncertainty ca. 1 %)

Find:

- The cross section area A in m^2 and in mm^2
- The current density J in A/m^2 and in A/mm^2
- The electric field magnitude E
- The potential difference between the two ends of the wire.
- The resistance R_0 of the wire at $T_0 = 293 \text{ K}$.

Pr 8.2 Electrical resistivity of Copper at different temperatures

At $T_0 = 293 \text{ K}$ we have for the electrical resistivity.

$\rho_{T_0} = 1,678 \cdot 10^{-8} \Omega \text{ m}$ (uncertainty ca. 1 %).

The temperature coefficient is $\alpha = 0,00393 \text{ K}^{-1}$.

Find:

- Please use the temperature coefficient to find the resistivity at $T = 400 \text{ K}$.
- Please find the deviation (the difference in %) relative to a table value based on experimental⁵ data at $T = 400 \text{ K}$, which is $\rho_T = 2,402 \cdot 10^{-8} \Omega \text{ m}$.

Test 8 Current

1i (5p) Please define the drift velocity $\vec{v}_d$ using $\vec{\ell}$ and Δt .

2. (20p) The current density as a vector $\vec{j}$.

- What is the direction?
- What is the magnitude?
- What is the flux of $\vec{j}$ through an area (expressed by an integral)

3i (5p) Write the relation between $\vec{j}$ and $\vec{v}_d$.

4i (20p) What is ?

- An ohmic material
- A non-ohmic material
- Please make a sketch and explain

5! (20p) Ohms law

- Write Ohms law using $\vec{E}$ and $\vec{j}$
- Derive the resistance R for a wire using ρ , L , and A .

6i (15p) Discharging a battery.

- Write the output voltage and output power
- Explain the terms.

7 (15p) Charging a battery.

- Write the input voltage and input power to the battery
- Explain the terms.

¹ www.allposters.com

² A. Matthiessen, C. Vogt: The influence of temperature on the electrical conductivity of alloys, *Ann. Phys. (Leipzig)*, 2, 122, 19-78 (1864).

³ R. A. Matula: Electrical resistivity of Copper, Gold, Palladium, and Silver. *J. Phys. Chem. Ref. Data*, Vol. 8, No. 4, (1979), see section 2.2.

⁴ R. A. Matula: Electrical resistivity of Copper, Gold, Palladium, and Silver. *J. Phys. Chem. Ref. Data*, Vol. 8, No. 4, (1979), see Table 2, T=293K

⁵ R. A. Matula: Electrical resistivity of Copper, Gold, Palladium, and Silver. *J. Phys. Chem. Ref. Data*, Vol. 8, No. 4, (1979), see Table 2, T=400K